ON THE MOTION OF A SATELLITE ABOUT ITS CENTER OF MASS UNDER THE ACTION OF GRAVITATIONAL MOMENTS

(O DVIZHENII SPUTNIKA OTNOSITEL'NO TSENTRA MASS POD DEISTVIEM GRAVITATSIONNYKH MOMENTOV)

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The motion of a satellite in an orbit sufficiently distant from the earth is mostly affected by the gravitational forces and moments. A number of papers by Beletskii, Sarychev [1-4] and others is devoted to the investigation of the motion of a satellite about its center of mass under the action of gravitational moments. In [1,3] Beletskii investigated the case when the kinetic energy of motion about the center of mass is much greater than the work of the external moments, the satellite possesses dynamic symmetry, and the orbit is circular or nearly circular. The motion of the satellite is in this case characterized by regular precession about the angular momentum vector as well as by a slow precession of the angular momentum vector itself.

Below are considered two cases of satellite motion under the action of gravitational moments when the existence of a small parameter permits application of the method of averaging and derivation of an asymptotic solution. The accuracy of these solutions is evaluated.

In the first case, the values of the three principal central moments of inertia of the satellite are assumed close although different. The orbital eccentricity and angular velocity of the satellite are arbitrary. Unperturbed motion represents steady rotation about the fixed axis. The translations of the fixed axis in space and relative to the body are determined.

In the second case, as well as in [1,3], the kinetic energy of the relative motion is considered large compared to the work of the gravitational moments, but no restrictions are imposed on the orbital

eccentricity and the moments of inertia. The satellite motion is constructed from the Euler-Poinsot motion about the angular momentum vector and the angular momentum vector motion itself. This vector performs precessional motion upon which are imposed finer effects: oscillations of the nutation and precession angles. The derived results generalize the corresponding formulas of Beletskii [1].

1. Let us consider the motion of a rigid body (satellite) in a central gravitational field. It may be considered that within the accuracy of terms up to the square of the ratio of the linear dimension of the satellite to that of the orbit, the motion of the satellite about its center of mass does not affect the motion of the center of the mass.

The center of mass moves on a Keplerian ellipse with eccentricity e and rotation period T_0. The dependence of the true anomaly ν on time t is given by the relation

$$\frac{d\nu}{dt} = \frac{\omega_0 (1 + e\cos\nu)^2}{(1 - e^2)^{3/2}} \qquad \left(\nu(t + T_0) = \nu(t) + 2\pi,\ \omega_0 = \frac{2\pi}{T_0}\right) \tag{1.1}$$

Let us introduce three right-handed Cartesian systems of coordinates, the origins of which will coincide with the center of inertia of the satellite.

The system of coordinates $x_1x_2x_3$ translates along with the center of inertia; the x_1-axis is parallel to the radius vector of the orbit perigee, the x_2-axis is parallel to velocity vector of the center of mass at perigee, and the x_3-axis is parallel to the normal to the orbit plane.

The y_3-axis of the system $y_1y_2y_3$ will be directed along the angular momentum vector $\mathbf{G}$ of the satellite about the center of inertia, the y_1-axis is perpendicular to y_3 and lies in the plane x_3y_3, while y_2 is perpendicular to y_1 and y_3 and, consequently, lies in the plane of the orbit x_1x_2 (Fig. 1).

The transfer from the orbital system of coordinates $x_1x_2x_3$ to the system $y_1y_2y_3$ is realized by two rotations: by the angle λ about x_3 and by the angle δ about y_2. The angles λ and δ define the orientation of the vector $\mathbf{G}$ in fixed space.

The axes of the coupled system of coordinates $z_1z_2z_3$ will coincide with the principal central axes of inertia of the satellite. Their orientation relative to the system of coordinates $y_1y_2y_3$ will be defined by the Euler angles θ, φ, ψ as well as by the direction cosines $\alpha_{ik} = y_iz_k$.

Unit vectors of the coordinates are denoted by the same letters as the axes. The following relationships are given between the direction cosines and the Euler angles

$$\alpha_{11} = \cos\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi$$
$$\alpha_{12} = -\sin\varphi\cos\psi - \cos\theta\cos\varphi\sin\psi$$
$$\alpha_{13} = \sin\theta\sin\psi, \qquad \alpha_{21} = \cos\varphi\sin\psi + \cos\theta\sin\varphi\cos\psi$$
$$\alpha_{22} = -\sin\varphi\sin\psi + \cos\theta\cos\varphi\cos\psi,$$
$$\alpha_{23} = -\sin\theta\cos\psi \tag{1.2}$$
$$\alpha_{31} = \sin\theta\sin\varphi, \qquad \alpha_{32} = \sin\theta\cos\varphi, \qquad \alpha_{33} = \cos\theta$$

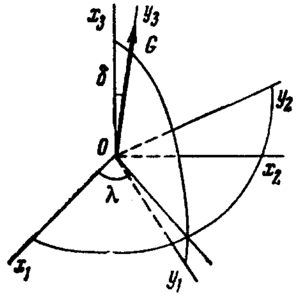

Fig. 1.

2. Let us write the equations of motion for the satellite about its center of inertia, choosing for the six required functions the angular momentum G and the angles δ, λ, θ, φ, ψ. The moment equation in terms of the y_1, y_2, y_3 components is

$$\frac{dG}{dt} = L_3, \qquad \frac{d\delta}{dt} = \frac{L_1}{G}, \qquad \frac{d\lambda}{dt} = \frac{L_2}{G\sin\delta} \tag{2.1}$$

where L_i are the projections of the moments of external forces about the center of inertia on the y_i-axes.

Projections of the $\boldsymbol{\omega}$ vector for the absolute angular velocity of the satellite on the z_1-, z_2-, z_3-axes are

$$\begin{aligned} p &= \dot\delta\alpha_{21} + \dot\lambda(\alpha_{31}\cos\delta - \alpha_{11}\sin\delta) + \dot\theta\cos\varphi + \dot\psi\alpha_{31} \\ q &= \dot\delta\alpha_{22} + \dot\lambda(\alpha_{32}\cos\delta - \alpha_{12}\sin\delta) - \dot\theta\sin\varphi + \dot\psi\alpha_{32} \\ r &= \dot\delta\alpha_{23} + \dot\lambda(\alpha_{33}\cos\delta - \alpha_{13}\sin\delta) + \dot\varphi + \dot\psi\alpha_{33} \end{aligned} \tag{2.2}$$

On the other hand, the projection of the vector $\mathbf{G}$ on these axes gives

$$G_1 = Ap = G\sin\theta\sin\varphi, \qquad G_2 = Bq = G\sin\theta\cos\varphi, \qquad G_3 = Cr = G\cos\theta \tag{2.3}$$

where A, B, C are the principal central moments of inertia of the satellite relative to the z_1-, z_2-, z_3-axes respectively.

Substituting p, q, r from the equation (2.3), $\dot\delta$, $\dot\lambda$ from (2.1), and α_{ij} from (1.2) into the equation (2.2), we obtain the solutions for the derivatives of the Euler angles θ, φ, ψ

$$\dot\theta = G\sin\theta\sin\varphi\cos\varphi\left(\frac{1}{A} - \frac{1}{B}\right) + \frac{L_2\cos\psi - L_1\sin\psi}{G}$$
$$\dot\varphi = G\cos\theta\left(\frac{1}{C} - \frac{\sin^2\varphi}{A} - \frac{\cos^2\varphi}{B}\right) + \frac{L_1\cos\psi + L_2\sin\psi}{G\sin\theta} \tag{2.4}$$

$$\dot{\psi} = G\left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right) - \frac{L_1\cos\psi + L_2\sin\psi}{G}\cot\theta - \frac{L_3}{G}\cot\delta$$

The equations (2.1) and (2.4) constitute an initial system of equations in the form convenient for application of asymptotic methods.

Let us write, in addition, using (2.3), (2.1) and (2.4), the expressions for the kinetic energy T of the satellite motion about its center of mass and its derivative

$$T = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) = \frac{G^2}{2}\left[\left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right)\sin^2\theta + \frac{\cos^2\theta}{C}\right] \quad (2.5)$$

$$\dot{T} = \frac{2T}{G}L_3 + G\sin\theta\left[\cos\theta\left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B} - \frac{1}{C}\right)(L_2\cos\psi - L_1\sin\psi) + \right.$$
$$\left. + \sin\varphi\cos\varphi\left(\frac{1}{A} - \frac{1}{B}\right)(L_1\cos\psi + L_2\sin\psi)\right] \quad (2.6)$$

3. We assume that the satellite is acted upon only by Newtonian forces directed toward a fixed center. Within the accuracy of higher order terms in the ratio of the linear dimensions of the satellite to those of the orbit, the moment of these forces about the center of mass of the satellite is [2]

$$\mathbf{L} = 3\mu R^{-3}\left[(C - B)\gamma_3\gamma_2\mathbf{z}_1 + (A - C)\gamma_1\gamma_3\mathbf{z}_2 + (B - A)\gamma_1\gamma_2\mathbf{z}_3\right] \quad (3.1)$$

Here γ_i are the direction cosines of the satellite inertia center radius vector $\mathbf{R}$ directed from the fixed center of attraction with the principal central axes of inertia z_i and μ is the gravitational constant. We project equation (3.1) on the y_i-axes. The direction cosines of the radius-vector $\mathbf{R}$ with the y_i-axes are denoted by β_i.

Expressing γ_i through β_i and α_{ik}, and μ, R through ν, e, ω_0 in accordance with the formulas for elliptic motion, we obtain

$$L_1 = 3\omega_0^2(1 + e\cos\nu)^3(1 - e^2)^{-3}\sum_{j=1}^{3}(\beta_2\beta_j s_{3j} - \beta_3\beta_j s_{2j})$$

$$L_2 = 3\omega_0^2(1 + e\cos\nu)^3(1 - e^2)^{-3}\sum_{j=1}^{3}(\beta_3\beta_j s_{1j} - \beta_1\beta_j s_{3j}) \quad (3.2)$$

$$L_3 = 3\omega_0^2(1 + e\cos\nu)^3(1 - e^2)^{-3}\sum_{j=1}^{3}(\beta_1\beta_j s_{2j} - \beta_2\beta_j s_{1j})$$

$$s_{ij} = A\alpha_{i1}\alpha_{j1} + B\alpha_{i2}\alpha_{j2} + C\alpha_{i3}\alpha_{j3} \quad (3.3)$$

For computing β_i we note that $\mathbf{R}$ lies in the orbit plane x_1x_2 and forms with the x_1-axis the angle ν. Then (see Fig. 1)

$$\beta_1 = \cos\delta\cos(\nu - \lambda), \qquad \beta_2 = \sin(\nu - \lambda), \qquad \beta_3 = \sin\delta\cos(\nu - \lambda) \quad (3.4)$$

4. In order to solve the system (2.1) and (2.5) we will apply the asymptotic method of Krylov and Bogoliubov (the method of averaging) [5] in the form developed by Volosov [6,7] (generalization of the method of the rapidly rotating phase of Bogoliubov and Zubarev).

Volosov [6,7] considered the system of the form

$$\dot{x} = \varepsilon X(x, y, t, \varepsilon), \qquad \dot{y} = Y(x, y, t, \varepsilon) \qquad (\varepsilon \ll 1) \qquad (4.1)$$

where x, X are n-dimensional, and y, Y are m-dimensional vector-functions, ε is a small parameter. The quantities x will be "slow", and y will be "rapid" variables. The general solution of the unperturbed (degenerate) system

$$x = \text{const}, \qquad \dot{y} = Y(x, y, t, 0) \qquad (4.2)$$

which is obtainable from (4.1) for $\varepsilon = 0$, is assumed known. We will denote this solution, which satisfies arbitrary initial conditions $y(t_0) = y_0$, by $y(x, y_0, t)$. The asymptotic solution of the system (4.1) in the kth approximation is sought in the form

$$\begin{aligned} x &= \xi + \varepsilon u_1(\xi, \eta, t) + \ldots + \varepsilon^{k-1} u_{k-1}(\xi, \eta, t) \\ y &= \eta + \varepsilon v_1(\xi, \eta, t) + \ldots + \varepsilon^{k-2} v_{k-2}(\xi, \eta, t) \end{aligned} \qquad (4.3)$$

where the variables ξ, η satisfy the system of kth approximation

$$\begin{aligned} \dot{\xi} &= \varepsilon A_1(\xi) + \varepsilon^2 A_2(\xi) + \ldots + \varepsilon^k A_k(\xi) \\ \dot{\eta} &= Y(\xi, \eta, t, 0) + \varepsilon B_1(\xi) + \ldots + \varepsilon^{k-1} B_{k-1}(\xi) \end{aligned} \qquad (4.4)$$

The function $A_1(\xi)$ is of the form

$$A_1(\xi) = M_t\{X(\xi, y, t, 0)\} \qquad (4.5)$$

where M_t denotes the averaging operation along the solutions of the unperturbed system (4.2)

$$M_t\{f(x, y, t)\} = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(x, y(x, y_0, t), t)\, dt \qquad (4.6)$$

It is assumed that the result of averaging in (4.5) is independent of the initial conditions t_0, y_0 which is valid for a wide class of cases.

References [6,7] give a construction algorithm for the functions u_i, v_i, A_i, B_i and formulate theorems substantiating this method. For certain general restrictions the difference of the kth approximation (4.3) from the exact solution will be for the variables x of order ε^k, and for the variables y of order ε^{k-1} on the interval of variation t of order ε^{-1}. Note that the system (4.4) is significantly simpler than the original one since the equations for the variables ξ are autonomous and are integrated separately.

5. Let the principal central moments of inertia for the satellite be nearly equal to each other, i.e. are of the form

$$A = J_0 + \varepsilon A', \qquad B = J_0 + \varepsilon B', \qquad C = J_0 + \varepsilon C' \qquad (\varepsilon \ll 1) \tag{5.1}$$

For $\varepsilon = 0$ it follows from (3.3) and (5.1) that $s_{ij} = J_0\delta_{ij}$ (δ_{ij} is the Kronecker delta) and then from (3.2) we have $L_1 = L_2 = L_3 = 0$. From the (2.1) and (2.4) we get for this case that G, δ, λ, θ, and φ are constant, and

$$\psi = GJ_0^{-1}t + \psi_0 \tag{5.2}$$

i.e. the satellite rotates uniformly about the translating axis of the angular momentum.

For small $\varepsilon \neq 0$, we obtain from (3.3) and (3.2) that $s_{ij} = J_0\delta_{ij} + O(\varepsilon)$, $L_i = O(\varepsilon)$. Then the system of seven equations (1.1), (2.1), (2.4) with (3.2) is apparently a system of the type (4.1), where the role of the "slow" variables (x) is played by G, δ, λ, θ, φ, while that of the "rapid" variables (y) is played by ψ and ν. In order to obtain the solution in the first approximation it is sufficient to simply average the right-hand sides of the equations (2.1) and (2.4) substituting for ν from the solution of the equation (1.1) and ψ from the equation (5.2). For fixed values of the "slow" variables, the right-hand sides of the equations, subject to averaging, will be the sums of terms of the form $f_1(\psi)\, f_2(\nu)$, where the functions f_1, f_2 are periodic in their arguments with periods 2π. Also, as may be easily shown, the Fourier expansion of $f_1(\psi)$ contains no harmonics higher than third. Therefore, the expansion of the right-hand sides of equations (2.1) and (2.4) into double Fourier series (in ψ and ν), after substitution of ψ and ν as functions of time, will be a sum of terms of the form

$$C_{mn} \cos [m (GJ_0^{-1} t + \psi_0)] \cos n\omega_0 t \qquad (m = 0, 1, 2, 3;\ n = 0, 1, 2 \ldots)$$

and similar terms where one or both cosine terms can be replaced by the sine terms. Let for no natural n no equality

$$G = nJ_0\omega_0, \qquad G = \frac{1}{2}nJ_0\omega_0, \qquad G = \frac{1}{3}nJ_0\omega_0 \tag{5.3}$$

be satisfied.

Then, the result of averaging the right-hand sides in accordance with (4.6) is independent of the initial value of ψ_0. In this case the time averaging can be replaced by the independent averaging with respect to ψ and ν, as with respect to the function of t.

If, however, any of the equalities (5.3) are satisfied, then resonance effects take place which are not considered here.

The time averaging of the functions dependent on ν is reduced, in accordance with (1.1), to the averaging with respect to ν as follows

$$M_t\{f(\nu)\} = \frac{1}{T_0}\int_0^{T_0} f(\nu)\,dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{(1-e^2)^{3/2} f(\nu)\,d\nu}{(1+e\cos\nu)^2} = (1-e^2)^{3/2} M_\nu\left\{\frac{f(\nu)}{(1+e\cos\nu)^2}\right\} \tag{5.4}$$

Averaging the right-hand sides of the equations (2.1) and (2.4) (taking into account (3.2), (3.3) and (1.2)) initially with respect to ψ, and then with respect to ν in accordance with (5.4), we will obtain the equations of first approximation in the form

$$\dot G = 0, \qquad \dot\delta = 0, \qquad \dot\psi = G\,J_0^{-1} + O(\varepsilon) \tag{5.5}$$

$$\dot\lambda = \frac{3\omega_0^2\cos\delta}{4G(1-e^2)^{3/2}}\{A + B + C - 3\,[(A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + C\cos^2\theta]\}$$

$$\dot\theta = G\sin\theta\sin\varphi\cos\varphi\left(\frac{1}{A} - \frac{1}{B}\right) + \frac{3\omega_0^2(1-3\cos^2\delta)}{4G(1-e^2)^{3/2}}(A-B)\sin\theta\sin\varphi\cos\varphi$$

$$\dot\varphi = G\cos\theta\left(\frac{1}{C} - \frac{\sin^2\varphi}{A} - \frac{\cos^2\varphi}{B}\right) - \frac{3\omega_0^2(1-3\cos^2\delta)}{4G(1-e^2)^{3/2}}\cos\theta\,(A\sin^2\varphi + B\cos^2\varphi - C)$$

For further simplification of (5.5) we note that within the accuracy of terms of order ε^2, in accordance with (5.1), we have

$$\frac{1}{A} = \frac{2}{J_0} - \frac{A}{J_0^2}, \qquad A = 2J_0 - \frac{J_0^2}{A}$$

and analogously for the moments of inertia B, C. Utilizing these approximate equalities we transform (5.5) without decreasing the order of accuracy in ε

$$\dot G = 0, \qquad \dot\delta = 0, \qquad \dot\psi = \omega + O(\varepsilon), \qquad \dot\lambda = \frac{3\omega_0^2\Phi\cos\delta}{4G(1-e^2)^{3/2}} \tag{5.6}$$

$$\dot\theta = G \sin\theta \sin\varphi \cos\varphi\left(\frac{1}{A} - \frac{1}{B}\right)D, \qquad \dot\varphi = G\cos\theta\left(\frac{1}{C} - \frac{\sin^2\varphi}{A} - \frac{\cos^2\varphi}{B}\right)D$$

Here (5.7)

$$\Phi = \frac{6TJ_0^2}{G^2} - J_0^2\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right), \quad D = 1 - \frac{3\omega_0^2(1 - 3\cos^2\delta)}{4\omega^2(1-e^2)^{3/2}}, \qquad \omega = \frac{G}{J_0}$$

where T is defined by the formula (2.5).

The solution of the system (5.6) approximates the exact solution of the system (2.1) and (2.4) in the interval of time of order $T_0\varepsilon^{-1}$ much higher than the satellite rotation period within the accuracy of order ε for "slow" variables G, δ, λ, θ, φ, and within the accuracy of order 1 for ψ.

The relative motion of the satellite described by the equations (5.6) separates into three parts: "rapid" motion (variable ψ) and two "slow" motions (variables θ, φ, and G, δ, λ).

The "rapid" motion represents the satellite rotation about the angular momentum vector with angular velocity $\omega = GJ_0^{-1}$, constant in accordance with the first equation in (5.6).

The equations for the variables θ, φ describe the motion of the angular momentum vector about the satellite. It is easy to see that these equations differ from the equations for θ, φ for the case of the free motion of the body (first two equations (2.4) for $\mathbf{L} = 0$, the case of Euler-Poinsot) only by the multiplier D (5.7), constant in the given approximation.

Thus, the action of the gravitational moments merely alters by a constant number D the translation velocity of the vector $\mathbf{G}$ along its trajectory of motion in the case of Euler-Poinsot. These trajectories are determined from the relationships stemming from (2.3) and (2.5) as

$$G_1^2 + G_2^2 + G_3^2 = G^2, \qquad \frac{G_1^2}{A} + \frac{G_2^2}{B} + \frac{G_3^2}{C} = 2T$$

where G and T are constant. A number of such trajectories for fixed T and various G is shown in Fig. 2, where the arrows give the direction of motion for the Euler-Poinsot case and where $A > B > C$ [8]. The quantity D (5.7) can have a wide range and even be negative which corresponds to the change in the direction of motion in Fig. 2. As in the case of Euler-Poinsot, the permanent rotation axes are the principal axes of inertia, the rotation about the z_2-axis being unstable while that about z_1, z_3 is stable.

However, when the condition

$$\omega = \frac{G}{J_0} = \sqrt{\frac{3(1 - 3\cos^2\delta)}{4(1 - e^2)^{3/2}}}\,\omega_0$$

is satisfied, the satellite rotation will be stationary for any orientation of the rotation axis relative to the satellite. In this case the moments of the centrifugal and gravitational forces in the first approximation are in equilibrium, and the vector **G** is not translating relative to the body.

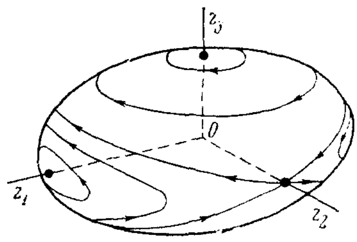

Fig. 2.

The motion of the vector **G** in space is described by the first three equations (5.6) and represents steady rotation of the vector **G** about the normal to the orbit plane at a constant angular displacement δ from it. The angular velocity of rotation $\dot{\lambda}$ is a quantity of order $\varepsilon\omega_0^{\,2}\omega^{-1}$ (since $\Phi \sim \varepsilon J_0$) while its sign depends on the character of the motion **G** relative to the satellite. For example, if $\delta < \pi/2$ then for rotation of the satellite about the axis of largest moment of inertia $z_1(G^2 = 2TA)$ we obtain from (5.7) that $\Phi < 0$, $\dot{\lambda} < 0$ (rotation of **G** in the direction opposite to the orbital motion), while for $G^2 = 2TC$ (rotation about z_3) $\dot{\lambda} > 0$.

6. Let us consider another case of small parameter introduction into the equations of motion of the satellite. The moments of inertia are considered arbitrary $(A \geqslant B \geqslant C)$. Let us assume that the angular velocity of the satellite relative motion is much higher than the angular velocity of the orbital motion, and let $\varepsilon \sim A\omega_0/G \ll 1$. Let the unit of time measurement be a quantity on the order of the period of relative motion, then $\omega_0 \sim \varepsilon$, while the moments of the gravitational forces $L_i \sim \varepsilon^2$ (3.2). The case being considered corresponds to large kinetic energy of rotation (compared to the work of the external forces); the asymptotics of such motions for a single degree of freedom system has been studied by Moiseev [9].

The unperturbed motion ($\varepsilon = 0$) will be the Euler-Poinsot motion, the quantities G, δ, λ and T being constant. The function ψ can be expressed in the form $\psi = \psi_1(t) + \psi_2(t)$, where θ, φ and ψ are periodic in t with a period τ of the motion of the vector **G** along the closed trajectories of Fig. 2 (or obtain constant increments 2π in time τ). The second component $\psi_2 = 2\pi t/\tau'$, while the periods τ and τ', dependent on G and T, are generally speaking, incommensurable [8].

In the perturbed motion ($\varepsilon \neq 0$) the "slow" variables (x) are G, δ, λ and T, while the "rapid" are φ and ψ (θ is expressed in terms of T

and φ by means of (2.5)). The equations of motion and (1.1) can be represented as

$$\dot{x} = \varepsilon^2 X(x, y, \nu), \qquad \dot{y} = Y_0(x, y) + \varepsilon^2 Y_1(x, y, \nu), \qquad \dot{\nu} = \varepsilon f(\nu) \quad (6.1)$$

It can be easily seen that in constructing the solutions of the type (4.3) for the system (6.1) $u_1 = v_1 \equiv 0$, while $A_1 = B_1 = A_3 \equiv 0$, $A_2 = M_t\{X\}$ in system (4.4). The solution for the "slow" variables will be sought in the form

$$x = \xi, \qquad \dot{\xi} = \varepsilon^2 A_2(\xi, \nu) = \varepsilon^2 M_t\{X(\xi, y, \nu)\} \quad (6.2)$$

neglecting the terms of order ε^2 in (4.3) and of order ε^4 in (4.4).

Therefore the error of the asymptotic solution for "slow" variables will be on the order of ε^2 on the interval of time of order ε^{-2}, which corresponds to the number of revolutions of the satellite in orbit of order ε^{-1} ($\Delta\nu \sim \varepsilon^{-1}$).

In order to construct the averaged system (6.2) it is necessary to average the right-hand sides of the equations of motion (for fixed "slow" variables and ν) along the Euler-Poinsot motion. These right-hand sides are periodic functions of θ, φ and ψ with periods 2π, while the periods τ and τ' are incommensurable. Therefore, using the arguments similar to those in Section 5, we establish that the averaging can be accomplished in two steps: with respect to θ, φ, ψ_1, and with respect to ψ_2, as with respect to the functions of time. Thus,

$$M_t\{f(\theta, \varphi, \psi)\} = \frac{1}{\tau\tau'}\int_0^{\tau}\int_0^{\tau'} f\left(\theta(t), \varphi(t), \ \psi_1(t) + \frac{2\pi t'}{\tau'}\right) dt'\, dt =$$

$$= \frac{1}{2\pi\tau}\int_0^{\tau}\int_0^{2\pi} f(\theta(t), \varphi(t), \psi)\, d\psi\, dt = M_1 M_\psi\{f(\theta, \varphi, \psi)\} \quad (6.3)$$

Here M_ψ denotes the averaging with respect to ψ, and M_1 in θ and φ related by (2.5) being carried out along the closed trajectories of the angular momentum vector in the Euler-Poinsot motion (Fig. 2).

Averaging the right-hand sides of equations (2.1) and (2.6) in accordance with (6.3) we obtain a system of first approximation

(6.4)

$$\dot{G} = 0, \qquad \dot{\delta} = -\frac{3\omega_0^2(1+e\cos\nu)^3}{2(1-e^2)^3 G}\beta_2\beta_3 N, \qquad \dot{\lambda} = \frac{3\omega_0^2(1+e\cos\nu)^3}{2(1-e^2)^3 G\sin\delta}\beta_1\beta_3 N$$

$$\dot{T} = \frac{3\omega_0^2(1+e\cos\nu)^3}{2(1-e^2)^3}(2\beta_3^2 - \beta_1^2 - \beta_2^2)\frac{(A-B)(B-C)(C-A)}{ABCG^2} M_1\{G_1 G_2 G_3\}$$

$$N = A + B + C - 3M_1\{(A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + C\cos^2\theta\} \quad (6.5)$$

On the strength of the symmetry of the vector **G** trajectory about the coordinate planes of the system $z_1z_2z_3$, it is obvious that $M_1(G_1, G_2, G_3) = 0$, and T = const. For nearly equal moments of inertia A, B, C (5.1) N coincides with Φ (5.7). In the general case, utilizing (2.3) and (2.5), we obtain

$$N = B - 2A - 2C + 6ACTG^{-2} + 3B(A-B)(B-C)G^{-2}M_1(q^2) \quad (6.6)$$

We substitute in (6.6) $q(t)$ from the Euler-Poinsot motion and average the function q^2 over its period. Finally, for the trajectory (Fig.2) of the vector **G** enveloping the axis $z_1(G^2 > 2TB)$, we have

$$N = B + C - 2A + 3\left(\frac{2TA}{G^2} - 1\right)\left[C + (B-C)\frac{K(k)-E(k)}{k^2K(k)}\right] \quad (6.7)$$

Here $K(k)$, $E(k)$ are complete elliptic integrals;

$$k^2 = \frac{(B-C)(2TA-G^2)}{(A-B)(G^2-2TC)} \quad (6.8)$$

For the trajectory of the vector **G** enveloping the axis $z_3(G^2 < 2TB)$, it is necessary to simply interchange A and C in the formulas (6.7) and (6.8).

The quantity N depends on the satellite moments of inertia and the relation G^2/T which determines the trajectory in Fig. 2 and is constant in the approximation considered. For satellite rotation about the axes $z_1(G^2 = 2TA)$, $z_3(G^2 = 2TC)$ we obtain from (6.7)

$$N = B + C - 2A < 0, \qquad N = B + A - 2C > 0$$

In case of dynamic symmetry ($A = B$) the formulas (6.6) and (2.5) give

$$N = 6ACTG^{-2} - A - 2C = (A-C)(2 - 3\sin^2\theta), \ \theta = \text{const} \quad (6.9)$$

In the considered approximation the relative motion of the satellite is composed of the Euler-Poinsot motion about the vector **G** (for constant G and T) and the motion of the vector **G** itself in space described by the equations (6.4) for δ, λ. Let us study these equations taking as the independent variable the true anomaly ν. Taking into account (1.1), (3.4) and (6.4), we write the equations for δ, λ in the form

$$\frac{d\delta}{d\nu} = \varkappa(1 + e\cos\nu)\sin\delta\sin(\lambda-\nu)\cos(\lambda-\nu) \quad \left(\varkappa = \frac{3\omega_0 N}{2(1-e^2)^{3/2}G}\right) \quad (6.10)$$

$$\frac{d\lambda}{d\nu} = \varkappa(1 + e\cos\nu)\cos\delta\cos^2(\lambda-\nu) \quad (6.11)$$

Apparently, the introduced dimensionless quantity κ is of order ε

and is constant in the considered approximation (in view of the constancy of G, T and N).

In the case of a circular orbit ($e = 0$) the equations (6.10) possess the first integral

$$\cos \delta + \tfrac{1}{2} \varkappa \sin^2 \delta \cos^2 (\lambda - \nu) = \text{const}$$

and their integration is reduced to quadratures.

It is simpler, however, to again apply the asymptotic methods for solving (6.10). As was indicated above, the introduced averaged equations determine the quantities δ, λ within the accuracy of order ε^2 (or κ^2) on the interval $\Delta\nu \sim \varepsilon^{-1} \sim \kappa^{-1}$. Therefore, it is sufficient to solve the equations (6.10) within this accuracy by finding the asymptotic solution in the second approximation in κ. The system (6.10) is in standard form [5]. Its solution in the second approximation is sought in the form

$$\delta = \xi + \varkappa u (\nu, \xi, \eta), \qquad \lambda = \eta + \varkappa v (\nu, \xi, \eta) \tag{6.12}$$

Determining the functions u, v by the known procedure [5], we obtain

$$u = \tfrac{1}{12} \sin \xi \ [3 \cos (2\nu - 2\eta) + 3e \cos (\nu - 2\eta) + e \cos (3\nu - 2\eta)] \tag{6.13}$$
$$v = \tfrac{1}{12} \cos \xi \ [3 \sin (2\nu - 2\eta) + 6e \sin \nu + 3e \sin (\nu - 2\eta) + e \sin (3\nu - 2\eta)]$$

The variables ξ, η satisfy the system of second approximation

$$\frac{d\xi}{d\nu} = \frac{1}{8} \varkappa^2 e^2 \sin \xi \cos \xi \sin 2\eta$$
$$\frac{d\eta}{d\nu} = \frac{1}{2} \varkappa \cos \xi + \frac{1}{16} \varkappa^2 (3 \cos^2 \xi - 1)\left(1 + \frac{2}{3} e^2 + e^2 \cos 2\eta\right) \tag{6.14}$$

Let us find within the required accuracy (error of order κ^2 on the interval of time of order κ^{-1}) the solution of system (6.14), satisfying the initial conditions $\xi(0) = \xi_0$, $\eta(0) = \eta_0$. It is easy to see that for such a solution in the interval $\Delta\nu \sim \kappa^{-1}$ the estimates

$$\xi - \xi_0 = O(\varkappa), \qquad \eta - \eta_0 - \tfrac{1}{2} \varkappa \nu \cos \xi_0 = O(\varkappa) \tag{6.15}$$

are valid which yield the solution of the system (6.14) in the first approximation. Transforming the right-hand sides of (6.14) with the aid of (6.15), and neglecting terms $O(\kappa^3)$ which includes errors in the solution of $O(\kappa^2)$, we have

$$\frac{d\xi}{d\nu} = \frac{1}{8} \varkappa^2 e^2 \sin \xi_0 \cos \xi_0 \sin (2\eta_0 + \varkappa \nu \cos \xi_0)$$

$$\frac{d\eta}{d\nu} = \frac{1}{2}\varkappa\cos\xi_0 - \frac{1}{2}\varkappa(\xi-\xi_0)\sin\xi_0 + \frac{1}{16}\varkappa^2(3\cos^2\xi_0 - 1)\times \times\left[1+\frac{2}{3}e^2 + e^2\cos(2\eta_0 + \varkappa\nu\cos\xi_0)\right] \quad (6.16)$$

Solutions of the system (6.16), satisfying the initial conditions, are

$$\xi = \xi_0 + \tfrac{1}{8}\varkappa e^2\sin\xi_0\,[\cos 2\eta_0 - \cos(2\eta_0 + \varkappa\nu\cos\xi_0)] \quad (6.17)$$

$$\eta = \eta_0 + \tfrac{1}{2}\varkappa\nu\cos\xi_0 + \tfrac{1}{16}\varkappa^2\nu\left[(3\cos^2\xi_0 - 1)\left(1+\tfrac{2}{3}e^2\right) - e^2\sin^2\xi_0\cos 2\eta_0\right] + + \tfrac{1}{8}\varkappa e^2\cos\xi_0\,[\sin(2\eta_0 + \varkappa\nu\cos\xi_0) - \sin 2\eta_0]$$

Substituting (6.13) and (6.17) into (6.12), we obtain

$$\delta = \xi_0 + \tfrac{1}{24}\varkappa\sin\xi_0\,\{6\cos[(2-\varkappa\cos\xi_0)\nu - 2\eta_0] + + 6e\cos[(1-\varkappa\cos\xi_0)\nu - 2\eta_0] + 2e\cos[(3-\varkappa\cos\xi_0)\nu - 2\eta_0] + + 3e^2\cos 2\eta_0 - 3e^2\cos(\varkappa\nu\cos\xi_0 + 2\eta_0)\}$$

$$\lambda = \eta_0 + \tfrac{1}{2}\varkappa\nu\cos\xi_0 + \tfrac{1}{16}\varkappa^2\nu\left[(3\cos^2\xi_0 - 1)\left(1+\tfrac{2}{3}e^2\right) - e^2\sin^2\xi_0\cos 2\eta_0\right] + + \tfrac{1}{24}\varkappa\cos\xi_0\,\{6\sin[(2-\varkappa\cos\xi_0)\nu - 2\eta_0] + 12e\sin\nu + + 6e\sin[(1-\varkappa\cos\xi_0)\nu - 2\eta_0] + 2e\sin[(3-\varkappa\cos\xi_0)\nu - 2\eta_0] + + 3e^2\sin(\varkappa\nu\cos\xi_0 + 2\eta_0) - 3e^2\sin 2\eta_0\} \quad (6.18)$$

This solution of the system (6.10) differs from the exact one by the quantities of order κ^2 in the interval $\Delta\nu \sim \kappa^{-1}$, and ξ_0 and η_0 are arbitrary constants. If one is limited to the accuracy of order κ, then the solution (6.18) simplifies and

$$\delta = \xi_0, \qquad \lambda = \eta_0 + {}^1/_2\varkappa\nu\cos\xi_0 \quad (6.19)$$

This solution of first approximation describes the rotation of the angular momentum vector, for constant δ, about the normal to the orbit plane with the velocity, in accordance with (6.11), given by

$$\dot{\lambda} = \frac{3\omega_0^2 N\cos\delta}{4G(1-e^2)^{3/2}} \quad (6.20)$$

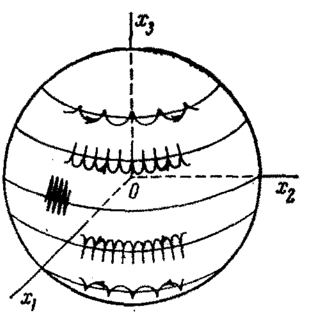

Fig. 3.

For $A = B$ and $e = 0$, utilizing the expression (6.9) for N, we obtain the formula derived by Beletskii [1].

The trajectories of the trace of the angular momentum vector on a unit sphere, fixed in the $x_1x_2x_3$ coordinate system, in the first approximation represent circles with δ = const. In the second approximation there appear oscillations of the angles δ and λ, and for increasing average values of δ (i.e. ξ_0) from 0 to $\pi/2$ the amplitude of δ oscillations increases while the amplitude of λ oscillations, as well as the average angular velocity of rotation, decreases. The total velocity $\dot{\lambda}$ for $\delta \neq \pi/2$ does not change sign and vanishes only at separate points where also $\dot{\delta} = 0$ (see (6.10)). For $\delta \sim \pi/2$ the variations in λ will be the quantities of second order compared to variations in δ. The trajectories of the vector **G** on a unit sphere, fixed relative to $x_1x_2x_3$, are shown in Fig. 3 with the effects indicated above, where $\kappa > 0$ (for $\kappa < 0$ only the direction of motion, indicated by arrows, changes along the trajectories).

On a circular orbit ($e = 0$) the oscillations of δ and λ are nearly sinusoidal with the angular frequency equal to twice the velocity of the orbital motion, and the curves in Fig. 3 are nearly cycloids subject to compression or extension along the axes of the coordinates. In the case of an elliptic orbit the oscillations of δ and λ become more complex: there appear the first and third harmonic as well as a substantial dependence of the form of oscillations upon the initial condition η_0, but the basic properties of the trajectories in Fig. 3 remain unaltered (particularly the existence of cusps directed towards the poles).

The regions of applicability for the asymptotic solutions in Sections 5 and 6, apparently, intersect: for $\omega \gg \omega_0$ (rapid relative motion) and nearly equal A, B, C, the results in Section 5 coincide with the first approximation in Section 6. Note, however, that the presented method is applicable for study of the rapid relative motion of a rigid body subject to moments of any nature.

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